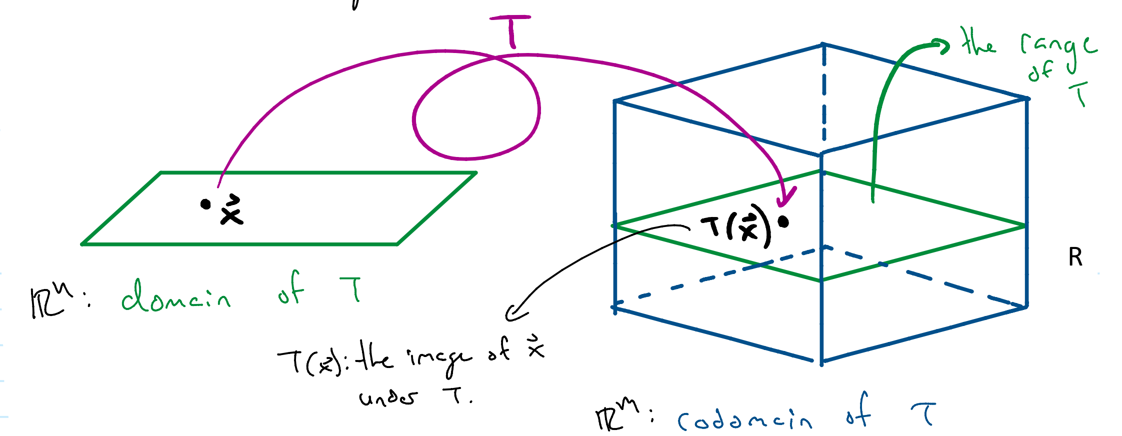
1.8 Linear Transformations (Intro.)

Wednesday, January 30, 2019 1:05 PM

"We now step "left" as it were to discuss a concept not immediately related to solving linear systems but central in linear algebra and some of it's applications: this idea, which we will motivate from matrix multiplication is that of the linear transformation."

Def: A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector $\vec{x}$ in $\mathbb{R}^n$ a vector $T(\vec{x})$ in $\mathbb{R}^m$.

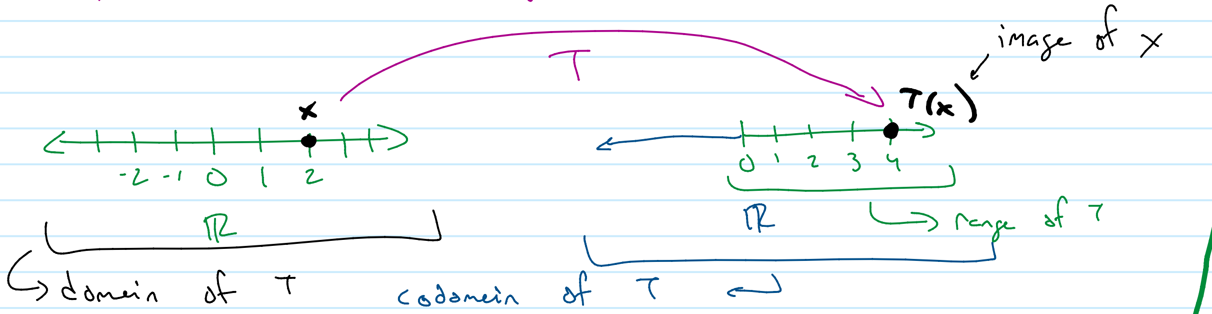
We call $\mathbb{R}^n$ the domain of T, $\mathbb{R}^m$ the codomain of T; $T(\vec{x})$ is the image of $\vec{x}$ in $\mathbb{R}^m$ and the range of T is the set of all images in $\mathbb{R}^m$.



Synonyms for a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ include:

"a function from $\mathbb{R}^n$ to $\mathbb{R}^m$" -or- "a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$"

Compare these terms: e.g. $f(x) = x^2$ is a transformation from $\mathbb{R}$ to $\mathbb{R}$



A familiar example: Matrix multiplication

Recall an $m \times n$ matrix A sends a vector $\vec{x}$ in $\mathbb{R}^n$ to a vector $A\vec{x}$ in $\mathbb{R}^m$. In this way, we see matrix multiplication gives a transformation:

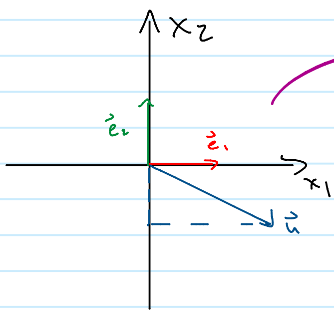
Ex/ If $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$ for every $\vec{x}$ in $\mathbb{R}^2$.

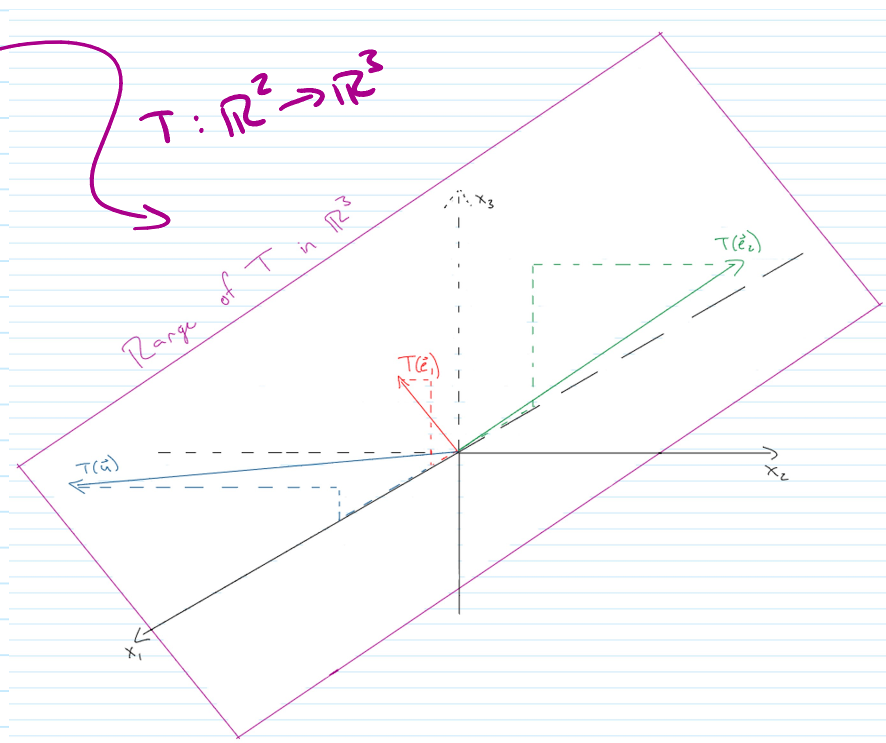
Explicitly, if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then

$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ -x_1 + 7x_2 \\ 3x_1 + 5x_2 \end{bmatrix} = A\vec{x}$$

We see the domain of T is $\mathbb{R}^2$, the codomain is $\mathbb{R}^3$ and the range of T is all linear combinations of the columns of A (i.e. $\text{span}\{\vec{a}_1, \vec{a}_2\}$ if $A = [\vec{a}_1 \ \vec{a}_2]$).

We sketch the image of a few vectors and the range of T.





$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{e}_1 - \vec{e}_2$

$$T(\vec{e}_1) = A\vec{e}_1 = \begin{bmatrix} 1 & -3 \\ -1 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$T(\vec{e}_2) = A\vec{e}_2 = \begin{bmatrix} 1 & -3 \\ -1 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

Notice an arbitrary vector in the

$= \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ $\quad T(\vec{e}_2) = A\vec{e}_2 = \begin{bmatrix} 1 & -3 \\ -1 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix}$

$$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 \\ -1 & 7 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} -3 \\ 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \\ 1 \end{bmatrix}$$

$$\vec{u} = 2\vec{e}_1 - \vec{e}_2 \implies T(\vec{u}) = 2T(\vec{e}_1) - T(\vec{e}_2)$$

Notice an arbitrary vector in the range of T is a linear combination of $T(\vec{e}_1)$ and $T(\vec{e}_2)$

(and an arbitrary vector in $\mathbb{R}^2$ is a linear combination of $\vec{e}_1, \vec{e}_2$)

We call such a transformation a matrix transformation

Linear Transformations

Def: A transformation T is linear if for all vectors $\vec{u}, \vec{v}$ in the domain of T and scalars c, d

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
2) $T(c\vec{u}) = cT(\vec{u})$

-or- $T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + T(\vec{v})$

Equivalently, a linear combination of inputs to T yields a linear combination of their respective outputs. (This is known as the superposition principle in applications.)

Note: Any transformation defined by a matrix is automatically linear:

$$T(c\vec{u}+d\vec{v}) = A(c\vec{u}+d\vec{v}) = A(c\vec{u}) + A(d\vec{v}) = c(A\vec{u}) + d(A\vec{v}) = cT(\vec{u}) + dT(\vec{v}).$$

Ex The dilation $T(\vec{x}) = r\vec{x}$ for some fixed $r \in \mathbb{R}$ is linear.

$$T(c\vec{u}+d\vec{v}) = r(c\vec{u}+d\vec{v}) = r(c\vec{u}) + r(d\vec{v})$$
$$= (rc)\vec{u} + (rd)\vec{v} = (cr)\vec{u} + (dr)\vec{v}$$
$$= c(r\vec{u}) + d(r\vec{v}) = cT(\vec{u}) + dT(\vec{v}).$$

Ex Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

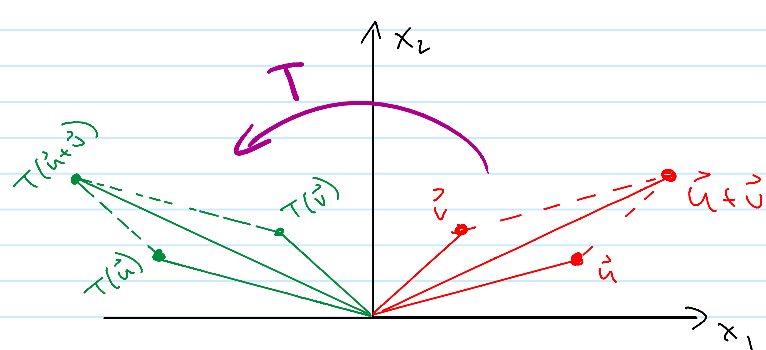
Let $\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and note $\vec{u}+\vec{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

Find $T(\vec{u})$, $T(\vec{v})$, $T(\vec{u}+\vec{v})$.

$$T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T(\vec{v}) = \cdots = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$T(\vec{u}+\vec{v}) = \cdots = \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$



Notice $T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v})$.

Geometrically: T is a rotation.

$\lfloor b \rfloor$.

Notice $T(\vec{u})+T(\vec{v}) = T(\vec{u}+\vec{v})$. __Geometrically__: T is a rotation.

This __should__ happen as T is a matrix transformation and therefore linear.